Higher Order Cumulant Truncation for Improved Moment Closure

Shreya Jha, Brook Eyob, Florian Schaefer

The Problem

We aim to understand how quantities involving randomness or complex dynamics evolve over time. This can be expressed as:

$$\frac{dX(t)}{dt} = f(X(t), t)$$

However, directly solving these high dimensional equations is computationally expensive. Instead, we use a reduced order model that evolves the **statistical moments**. The n-th moment is defined as:

$$\mu_n = \mathbb{E}[X^n] = \int x^n p(x)\, dx$$

Deriving **moment evolution equations** from the original system results in an **infinite hierarchy of equations**:

$$\frac{d\mu_1}{dt} = H(\mu_1, \mu_2, \dots)$$
$$\frac{d\mu_2}{dt} = H(\mu_2, \mu_3, \dots)$$

Gaussian Closure

A common way to close the system is by assuming the random variables follow a **Gaussian distribution**. In this case, moments of order $n > 2$ are approximated as functions of the first two moments:

$$\mu_n \approx f(\mu_1, \mu_2)$$

The Gaussian closure is used since it is the only distribution where higher order moments can be expressed using the first two moments. Although this closure method is widely used, it ignores higher order information that might be useful.

Cumulant Truncation

We approximate higher-order moments by truncating **cumulants**, κ_n, an alternative to moments capturing the same information. Based on the **Central Limit Theorem**, higher-order cumulants decay faster. For m identical random variables X:

$$\kappa_n\left(\frac{1}{m}\sum X_i\right) = \frac{1}{m^{n-1}}\kappa_n(X)$$

This shows that higher-order cumulants decay to zero, allowing for the truncation of cumulants beyond a certain order without significant loss of information. By choosing how much higher-order information to retain, this truncation allows for system closure. We have developed techniques based on this.

Naive Truncation

Naive truncation is a method that involves setting higher-order cumulants to zero and using this approximation to convert back to moments. Specifically:

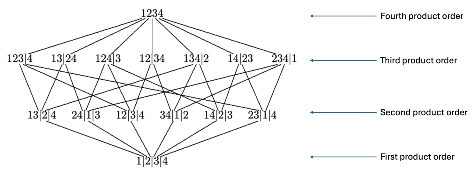
$$(\mu_1, \mu_2, \dots, \mu_k) \to (\kappa_1, \kappa_2, \dots, \kappa_k)$$
$$(\kappa_1, \dots, \kappa_k) \to (\kappa_1, \kappa_2, \dots, \kappa_k, 0_{k+1}, \dots, 0_l)$$
$$(\kappa_1, ., \kappa_k, 0_{k+1}, ., 0_l) \to (\mu_1, ., \mu_k, \mu'_{k+1}, ., \mu'_l)$$

Product Order Truncation

Moments are computed as the sum of the products of lower order cumulants.

$$\mu_n = \sum_{k=1}^{n}\prod_{j=1}^{k}\kappa_j$$

The **product order truncation** method truncates cumulants based on the order of their products, not individual cumulants.

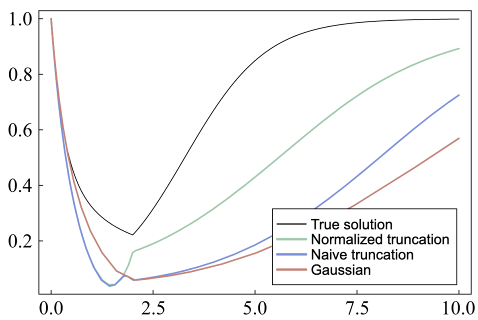


Normalized Truncation

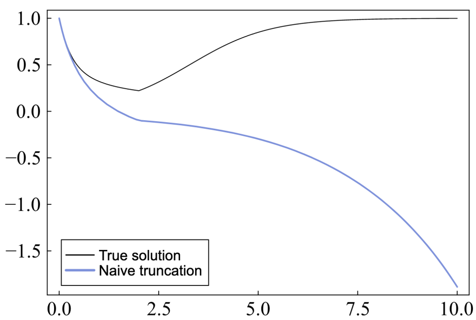
Normalized truncation sets the first cumulant order (mean) to 0 and the second cumulant order (variance) to 1, adjusting the others accordingly. This simplifies computations by not accounting for the mean.

$$\kappa\left(\frac{1}{\sqrt{N}}\sum X_i - \bar{X}\right)$$
$$= \left(0, \kappa_{ij}, \frac{1}{\sqrt{N}}\kappa_{ijk}, \frac{1}{N}\kappa_{ijkl}, \dots\right)$$

Truncation Results



In certain cases, naive and normalized truncation outperform the Gaussian method, with normalized truncation showing the most improvement.



However, in other cases, we see that these cumulant truncation methods exhibit unstable behavior. This is due to the violation of this property:

$$X > 0 \implies \mathbb{E}[X] > 0$$

Central Limiting Method

The Central Limiting Method (CLT) addresses the instability issues with a natural decay instead of abrupt truncation. It first converts the moments μ_n to cumulants κ_n, then applies the decay, and then converts the adjusted cumulants back to moments:

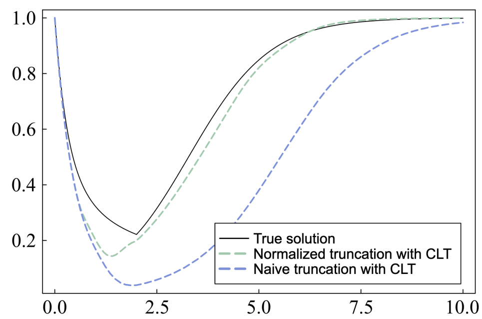
$$\kappa_n = f(\mu_1, \mu_2, \dots, \mu_n)$$
$$\kappa'_n = \kappa_n \exp(-\alpha n)$$
$$\mu'_n = g(\kappa'_1, \kappa'_2, \dots, \kappa'_n)$$

Sum of Squares Projection

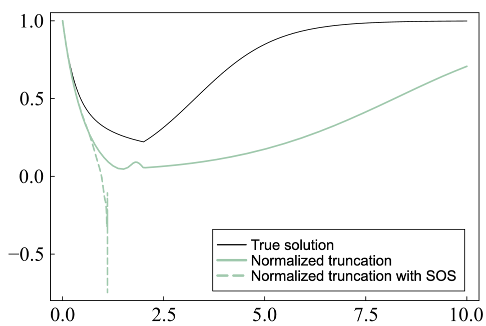
Moment sequences that have real representing measures correspond to the dual set of positive polynomials. The dual set of the sums of squares (SOS) polynomial consists of sequences with a positive semi-definite moment matrix. We project those without a representing measure into the SOS dual.

$$\Sigma^* = \{L(p^2) \geq 0\, \forall p \in R[x]\}$$
$$M_{\succeq} = \{y \in R^{N_n} \mid M(y) \succeq 0\} = \Sigma^*$$

CLT and SOS Results



Applying the CLT method prevents the instabilities that normalized and naive truncation were suffering with.



The SOS method, however, fails to prevent instabilities and, in fact, introduces them in cases where they previously did not occur.

Future Work

We observed improved performance using the naive and normalized truncation methods, especially with the added stability from the CLT-based approach. However, the SOS Projection has yet to show the expected stability gains. Future work will focus on refining the SOS Projection to enhance moment sequence stability.